to be equal to zero. We can carry out the integration in (2.9) for discrete values of $m = -2n$ $(n = 0, 1, 2, \ldots)$. Let us write out the expressions obtained for the first three values of m

$$u = k\{x + b[1 - H(m, \eta)]\}$$
$$H(0, \eta) = [\eta \operatorname{Ei}(-\eta) + e^{-\eta}]/[\operatorname{Ei}(-1) + e^{-1}]$$
$$H(-2, \eta) = e^{2(1-\eta)}, \quad H(-4, \eta) = {}^1/_5(3\eta + 2)\, e^{3(1-\eta)}$$

where $\operatorname{Ei}(z)$ is an integral exponential function.

The solutions discussed above can also be used in a situation when the surface of the cylinder not only stretches, but also moves with constant velocity in the direction of the x axis. In this case we replace the boundary conditions (1.1) by the relations

$$r = R, \quad u = kx + U_w, \quad v = 0 \tag{2.10}$$

Solved (2.4) satisfies this condition at another value of the constant A:

$$A = {}^5/_3 - 3\ln 3 - {}^9/_2 U_w/U_\infty$$

The solution (2.6)-(2.8) can be generalized to the case (2.10), provided that we add the term $\rho k U_w c^2 x$, to p in (2.8) and the term $\sqrt{\nu k} g$ to u, where

$$g = U_w c (\nu k)^{-1/2} [1 + (\lambda - 3)\, e^{\lambda(1-\eta)}]$$

The solution (2.9) can be generalized to the case (2.10) by leaving the constant A undetermined.

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ON THE PROBLEM OF THE COLLAPSE OF CAVITATIONAL VOIDS*

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The part played by the capillary properties of a medium in the problem of the collapse on an empty spherical cavity in a viscous incompressible fluid modelling the stage of collapse of cavitational voids is studied. Methods of qualitative theory are used to study the differential equations describing the dynamics of the boundary of the cavity. A pattern of behaviour of the integral curves in the phase plane is obtained and used to produce a complete description of all possible modes of collapse of the cavity.

The problem of the filling of an empty spherical cavity with an ideal incompressible fluid was studied by Rayleigh /1/, who showed that the velocity of the liquid boundary of the cavity increases without limit as $R^{-3/2}$ as its radius R decreases to zero. The time in which the cavity disappears is always finite.

Taking into account the viscosity of the fluid /2/ leads to the conclusion that a critical Reynolds number Re^* exists, separating two, essentially different modes of filling the cavity. When $\mathrm{Re} > \mathrm{Re}^*$, the character of the motion is analogous to that in Rayleigh's case. The principal term of the expansion of the velocity V of the boundary of the cavity

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is equal to $CR^{-3/2}$ as $R \to 0$, but the value of the constant C is smaller than that in /1/. When $\mathrm{Re} < \mathrm{Re}^*$, there is no cumulative effect, the fluid near the focus is retarded in accordance with the rule $V \sim R$, and the time of closing the cavity becomes infinitely long. The intermediate case of $\mathrm{Re} = \mathrm{Re}^*$ corresponds, as $R \to 0$, to an increase in the modulus of $V \sim R^{-1}$ and a finite time of the closure of the cavity.

Actually, since the boundary of the cavity is also an interphase boundary, it follows that it has a certain surface tension σ. Let us consider the Rayleigh problem, taking into account the above factor and the viscosity of the fluid assuming, as in /1, 2/, that the cavity is empty. When the problem is formulated in this manner, it can be applied to modelling the process of collapse of the cavitational bubbles /3/ and merits attention for this reason. It should be noted that a numerical investigation /3, 4/ has indicated the existence of several modes of collapse. However, even this approach did not reveal the general laws inherent in this class of motions. The purpose of this paper is to give a complete elucidation of these relationships.

Under the assumptions made above, the dynamics of the boundary of a collapsing bubble can be described by the following Cauchy problem in dimensionless variables:

$$ux\frac{du}{dx} + \frac{3}{2}u^2 + 4\frac{u}{x} + 2\frac{\delta}{x} + 1 = 0, \quad u(\mathrm{Re}) = 0 \tag{1}$$

$$u = V\sqrt{\frac{\rho}{p_\infty}}, \quad x = \frac{R}{R_0}\mathrm{Re}, \quad \mathrm{Re} = \frac{R_0}{\mu}\sqrt{\rho p_\infty}, \quad \delta = \frac{\sigma}{\mu}\sqrt{\frac{\rho}{p_\infty}}$$

Here R_0 is the initial radius of the bubble, p_∞ is the pressure in the fluid at infinity, ρ is the density, μ is the dynamic viscosity and Re is the Reynolds number.

The number δ, equal to the product of the Weber and Reynolds numbers of the problem, characterizes the balance of the surface forces and viscous forces at the boundary of the cavity.

Making the substitution $y = u^{-1}$, we reduce Eq.(1) to the form

$$\frac{dy}{dx} = \frac{y}{x}\left[\frac{3}{2} + 4\frac{y}{x} + 2\delta\frac{y^2}{x} + y^2\right] \tag{2}$$

and the region

$$y \leqslant 0, \ x \geqslant 0 \tag{3}$$

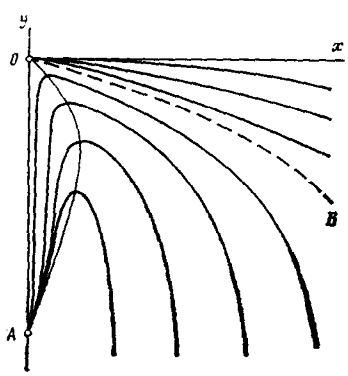
of the x, y-plane is the only region possessing any physical meaning during the period of collapse.

Assuming that the Reynolds number is finite (a viscous fluid), we shall investigate the influence of the number δ (the surface tension) on the closure of the spherical cavity.

In the case when $\delta = 0$ $(\sigma = 0)$, discussed in /2/, Eq.(2) has a unique finite singularity $O(0, 0)$ of a complex (saddle-node) character, and the neighbourhood of the point O belonging to the region (3) contains a single sector of each type. The separatrix of the point O separating them, corresponds to the critical value $\mathrm{Re} = \mathrm{Re}^*$, and the integral curve of Eq. (2) belongs to the node (saddle) sector when $\mathrm{Re} > \mathrm{Re}^*$ $(\mathrm{Re} < \mathrm{Re}^*)$. The character of the motion of the fluid in each of the above three cases has already been discussed.

When $\delta > 0$ $(\sigma \neq 0)$, a second complex singularity $A(0, -2/\delta)$, appears in Eq.(2), and the character of the singularity at the point O is identical, as can easily be shown, to the case $\delta = 0$.

Let us investigate the type of singular point A. Making the change of variables



$$\zeta = a(y - y_A) + bx, \quad y_A = -2/\delta$$
$$a = -4y_A, \quad b = y_A(3/2 + y_A^2)$$

we reduce Eq.(2) to the canonical form /5/

$$d\zeta/dx = (a\zeta + \varphi(x, \zeta))/x^2 \tag{4}$$

where the expansion of φ near zero begins with terms of at least second order. From the form of Eq.(4) we see that the singularity in question is the simplest saddle-node /5/, containing two saddle sectors and a single node sector. By virtue of the condition $a > 0$, the hyperbolic regions are situated in the half-plane $x < 0$ and have, therefore, no physical meaning. In the parabolic region $(x > 0)$, all integral curves enter the singular point at zero inclination (they are tangent to the axis $\zeta = 0$).

The phase pattern of the initial Eq.(2) at $\delta > 0$ in the physically interesting part of the x, y-plane, is shown in the figure. As we have already said, the nature of the singularity at the point O does not depend on the quantity δ, and hence the inclination of the separatrix OB at the zero is always equal to $^1/_8$, just as in /2/. Moving along the separatrix to $u=0$, we arrive at some critical value of the Reynolds number Re* depending, in this case, on δ. The dependence Re* (δ), in the interval $0 \leqslant \delta \leqslant 1$ typical for practical applications, constructed from the numerical results of solving Eqs.(2) and (1) by the fourth order Runge-Kutta method, is nearly linear $\mathrm{Re}^* = 8.7 - 1.55\,\delta$ (with an error of less than 1%).

The integral curves lying above the curve OB correspond to the numbers $\mathrm{Re} > \mathrm{Re}^*$. All these curves enter the point O touching the zero isocline of Eq.(2), which is the x axis. The second zero isocline (curve OA in the figure) is described by the equation

$$x = -2y\,(2 + \delta y)/(^3/_2 + y^2)$$

All integral curves situated below the separatrix OB (they correspond to $\mathrm{Re} < \mathrm{Re}^*$), as well as the line OA, enter the point A at the same inclination $k = -b/a$.

The nature of the behaviour of the solutions of (2) implies the existence of three different models of the collapse of the bubble.

When $\mathrm{Re} > \mathrm{Re}^*$ and $\mathrm{Re} = \mathrm{Re}^*$, the modulus of the velocity $|V|$ increases without limit as $R \to 0$, and its order of magnitude is equal to $R^{-3/2}$ and R^{-1} respectively. Both these cases are qualitatively equivalent to the corresponding cases in /2/.

The part played by σ is essential for the motions corresponding to the numbers $\mathrm{Re} < \mathrm{Re}^*$. The fact that in these cases all solutions belong to the nodal region of the point A, implies the finiteness (and also the non-zero value) of the velocity V at the instant of closure of the bubble, and its single value $V_A = -\sigma/\mu$ for all trajectories of the family in question. (For example, in the case of water we have $\sigma \approx 73 \cdot 10^{-3}$ N/m, $\mu = 10^{-3}$ N·sec/m^2 and $|V_A| \approx 73$ m/sec). For this reason, the time of collapse of the bubble is always finite. We also note the non-monotonic nature of the change in the velocity of the bubble boundary when $\mathrm{Re} < \mathrm{Re}^*$. In this case a maximum value $|V|$ always exists, determined by the point of intersection of the corresponding integral curve with the isocline OA.

Thus when σ and μ are fixed, the value of the velocity V at the instant $R=0$ does not depend on the initial radius of the bubble (nor on p_∞ and ρ), provided that it is less than some critical radius R_0^* calculated using the value of Re*. The values of p_∞ and ρ affect only the quantity R_0^*, i.e. the upper limit of the size of the bubbles possessing the above property.

The quantitative estimation of the critical size of the bubble in water and glycerine yields, in the case in question, using the data obtained in /2/, values close to those obtained in /2/, by virtue of the weak dependence of Re* on δ.

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